一般相対論 練習問題解答

練習問題 1

$$I \equiv e^{2\phi}\left(\frac{dt}{d\tau}\right)^2 - e^{2\lambda}\left(\frac{dr}{d\tau}\right)^2 - \chi^2\left(\frac{d\theta}{d\tau}\right)^2 - \chi^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2$$

(1.1)

$$\frac{\partial I}{\partial t} = 2\dot\phi e^{2\phi}\left(\frac{dt}{d\tau}\right)^2 - 2\dot\lambda e^{2\lambda}\left(\frac{dr}{d\tau}\right)^2 - 2\dot\chi\chi\left(\frac{d\theta}{d\tau}\right)^2 - 2\dot\chi\chi\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2$$

$$\begin{aligned}\frac{d}{d\tau}\frac{\partial I}{\partial(dt/d\tau)} &= \frac{d}{d\tau}\left(2e^{2\phi}\frac{dt}{d\tau}\right)\\ &= 4\dot\phi e^{2\phi}\left(\frac{dt}{d\tau}\right)^2 + 4\phi' e^{2\phi}\frac{dt}{d\tau}\frac{dr}{d\tau} + 2e^{2\phi}\frac{d^2t}{d\tau^2}\end{aligned}$$

$$\begin{aligned}2e^{2\phi}\frac{d^2t}{d\tau^2} + 2\dot\phi e^{2\phi}\left(\frac{dt}{d\tau}\right)^2 + 4\phi' e^{2\phi}\frac{dt}{d\tau}\frac{dr}{d\tau} + 2\dot\lambda e^{2\lambda}\left(\frac{dr}{d\tau}\right)^2&\\ +2\dot\chi\chi\left(\frac{d\theta}{d\tau}\right)^2 + 2\dot\chi\chi\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2 &= 0\end{aligned}$$

$$\begin{aligned}\frac{d^2t}{d\tau^2} + \dot\phi\left(\frac{dt}{d\tau}\right)^2 + 2\phi'\frac{dt}{d\tau}\frac{dr}{d\tau} + \dot\lambda e^{2\lambda-2\phi}\left(\frac{dr}{d\tau}\right)^2&\\ +\dot\chi\chi e^{-2\phi}\left(\frac{d\theta}{d\tau}\right)^2 + \dot\chi\chi e^{-2\phi}\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2 &= 0\end{aligned}$$

$\dfrac{dt}{d\tau}\dfrac{dr}{d\tau}$ の係数は、${\Gamma^t}_{tr} + {\Gamma^t}_{rt} = 2{\Gamma^t}_{tr}$ であることに注意。

$${\Gamma^t}_{tt} = \dot\phi,\quad {\Gamma^t}_{tr} = {\Gamma^t}_{rt} = \phi',\quad {\Gamma^t}_{rr} = \dot\lambda e^{2\lambda-2\phi},\quad {\Gamma^t}_{\theta\theta} = \dot\chi\chi e^{-2\phi},\quad {\Gamma^t}_{\varphi\varphi} = \dot\chi\chi e^{-2\phi}\sin^2\theta$$

(1.2)

$$\frac{\partial I}{\partial r} = 2\phi' e^{2\phi}\left(\frac{dt}{d\tau}\right)^2 - 2\lambda' e^{2\lambda}\left(\frac{dr}{d\tau}\right)^2 - 2\chi'\chi\left(\frac{d\theta}{d\tau}\right)^2 - 2\chi'\chi\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2$$

$$\begin{aligned}\frac{d}{d\tau}\frac{\partial I}{\partial(dr/d\tau)} &= \frac{d}{d\tau}\left(-2e^{2\lambda}\frac{dr}{d\tau}\right)\\ &= -4\dot{\lambda}e^{2\lambda}\frac{dt}{d\tau}\frac{dr}{d\tau} - 4\lambda' e^{2\lambda}\left(\frac{dr}{d\tau}\right)^2 - 2e^{2\lambda}\frac{d^2 r}{d\tau^2}\end{aligned}$$

$$\begin{aligned}2e^{2\lambda}\frac{d^2r}{d\tau^2} + 2\phi' e^{2\phi}\left(\frac{dt}{d\tau}\right)^2 + 4\dot{\lambda}e^{2\lambda}\frac{dt}{d\tau}\frac{dr}{d\tau} + 2\lambda' e^{2\lambda}\left(\frac{dr}{d\tau}\right)^2&\\ -2\chi'\chi\left(\frac{d\theta}{d\tau}\right)^2 - 2\chi'\chi\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2 &= 0\end{aligned}$$

$$\begin{aligned}\frac{d^2r}{d\tau^2} + \phi' e^{2\phi-2\lambda}\left(\frac{dt}{d\tau}\right)^2 + 2\dot{\lambda}\frac{dt}{d\tau}\frac{dr}{d\tau} + \lambda'\left(\frac{dr}{d\tau}\right)^2&\\ -\chi'\chi e^{-2\lambda}\left(\frac{d\theta}{d\tau}\right)^2 - \chi'\chi e^{-2\lambda}\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2 &= 0\end{aligned}$$

$$\Gamma^r{}_{tt} = \phi' e^{2\phi-2\lambda},\quad \Gamma^r{}_{tr} = \Gamma^r{}_{rt} = \dot{\lambda},\quad \Gamma^r{}_{rr} = \lambda'$$

$$\Gamma^r{}_{\theta\theta} = -\chi'\chi e^{-2\lambda},\quad \Gamma^r{}_{\varphi\varphi} = -\chi'\chi e^{-2\lambda}\sin^2\theta$$

(1.3)

$$\frac{\partial I}{\partial\theta} = -2\chi^2\sin\theta\cos\theta\left(\frac{d\varphi}{d\tau}\right)^2$$

$$\begin{aligned}\frac{d}{d\tau}\frac{\partial I}{\partial(d\theta/d\tau)} &= \frac{d}{d\tau}\left(-2\chi^2\frac{d\theta}{d\tau}\right)\\ &= -4\dot{\chi}\chi\frac{dt}{d\tau}\frac{d\theta}{d\tau} - 4\chi'\chi\frac{dr}{d\tau}\frac{d\theta}{d\tau} - 2\chi^2\frac{d^2\theta}{d\tau^2}\end{aligned}$$

$$2\chi^2\frac{d^2\theta}{d\tau^2} + 4\dot{\chi}\chi\frac{dt}{d\tau}\frac{d\theta}{d\tau} + 4\chi'\chi\frac{dr}{d\tau}\frac{d\theta}{d\tau} - 2\chi^2\sin\theta\cos\theta\left(\frac{d\varphi}{d\tau}\right)^2 = 0$$

$$\frac{d^2\theta}{d\tau^2} + 2\frac{\dot{\chi}}{\chi}\frac{dt}{d\tau}\frac{d\theta}{d\tau} + 2\frac{\chi'}{\chi}\frac{dr}{d\tau}\frac{d\theta}{d\tau} - \sin\theta\cos\theta\left(\frac{d\varphi}{d\tau}\right)^2 = 0$$

$$\Gamma^\theta{}_{t\theta} = \Gamma^\theta{}_{\theta t} = \frac{\dot{\chi}}{\chi},\quad \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{\chi'}{\chi},\quad \Gamma^\theta{}_{\varphi\varphi} = -\sin\theta\cos\theta$$

(1.4)

$$\frac{\partial I}{\partial \varphi} = 0$$

$$\begin{aligned}\frac{d}{d\tau}\frac{\partial I}{\partial(d\varphi/d\tau)} &= \frac{d}{d\tau}\left(-2\chi^2\sin^2\theta\frac{d\varphi}{d\tau}\right) \\ &= -4\dot{\chi}\chi\sin^2\theta\frac{dt}{d\tau}\frac{d\varphi}{d\tau} - 4\chi'\chi\sin^2\theta\frac{dr}{d\tau}\frac{d\varphi}{d\tau} - 2\chi^2\sin^2\theta\frac{d^2\varphi}{d\tau^2}\end{aligned}$$

$$2\chi^2\sin^2\theta\frac{d^2\varphi}{d\tau^2} + 4\dot{\chi}\chi\sin^2\theta\frac{dt}{d\tau}\frac{d\varphi}{d\tau} + 4\chi'\chi\sin^2\theta\frac{dr}{d\tau}\frac{d\varphi}{d\tau} + 4\chi^2\sin\theta\cos\theta\frac{d\theta}{d\tau}\frac{d\varphi}{d\tau} = 0$$

$$\frac{d^2\varphi}{d\tau^2} + 2\frac{\dot{\chi}}{\chi}\frac{dt}{d\tau}\frac{d\varphi}{d\tau} + 2\frac{\chi'}{\chi}\frac{dr}{d\tau}\frac{d\varphi}{d\tau} + 2\cot\theta\frac{d\theta}{d\tau}\frac{d\varphi}{d\tau} = 0$$

$$\Gamma^{\varphi}{}_{t\varphi} = \Gamma^{\varphi}{}_{\varphi t} = \frac{\dot{\chi}}{\chi}, \quad \Gamma^{\varphi}{}_{r\varphi} = \Gamma^{\varphi}{}_{\varphi r} = \frac{\chi'}{\chi}, \quad \Gamma^{\varphi}{}_{\theta\varphi} = \Gamma^{\varphi}{}_{\varphi\theta} = \cot\theta$$

Γ	t	r	θ	φ
tt	$\dot{\phi}$	$\phi' e^{2\phi-2\lambda}$	0	0
tr	ϕ'	$\dot{\lambda}$	0	0
$t\theta$	0	0	$\dot{\chi}/\chi$	0
$t\varphi$	0	0	0	$\dot{\chi}/\chi$
rt	ϕ'	$\dot{\lambda}$	0	0
rr	$\dot{\lambda}e^{2\lambda-2\phi}$	λ'	0	0
$r\theta$	0	0	χ'/χ	0
$r\varphi$	0	0	0	χ'/χ
θt	0	0	$\dot{\chi}/\chi$	0
θr	0	0	χ'/χ	0
$\theta\theta$	$\dot{\chi}\chi e^{-2\phi}$	$-\chi'\chi e^{-2\lambda}$	0	0
$\theta\varphi$	0	0	0	$\cot\theta$
φt	0	0	0	$\dot{\chi}/\chi$
φr	0	0	0	χ'/χ
$\varphi\theta$	0	0	0	$\cot\theta$
$\varphi\varphi$	$\dot{\chi}\chi e^{-2\phi}\sin^2\theta$	$-\chi'\chi e^{-2\lambda}\sin^2\theta$	$-\sin\theta\cos\theta$	0

練習問題 2

$$R_{\alpha\beta} = \partial_\mu \Gamma^\mu{}_{\alpha\beta} - \partial_\beta \Gamma^\mu{}_{\alpha\mu} + \Gamma^\mu{}_{\gamma\mu}\Gamma^\gamma{}_{\alpha\beta} - \Gamma^\mu{}_{\gamma\beta}\Gamma^\gamma{}_{\alpha\mu}$$

(2.1)

$$\begin{aligned}
\partial_\mu \Gamma^\mu{}_{tt} &= \partial_t \Gamma^t{}_{tt} + \partial_r \Gamma^r{}_{tt} \\
&= \partial_t \dot{\phi} + \partial_r \phi' e^{2\phi-2\lambda} \\
&= \ddot{\phi} + (\phi'' + 2\phi'^2 - 2\phi'\lambda')e^{2\phi-2\lambda}
\end{aligned}$$

$$\begin{aligned}
\partial_t \Gamma^\mu{}_{t\mu} &= \partial_t(\dot{\phi} + \dot{\lambda} + 2\dot{\chi}/\chi) \\
&= \ddot{\phi} + \ddot{\lambda} + 2\ddot{\chi}/\chi - 2\dot{\chi}^2/\chi^2
\end{aligned}$$

$$\begin{aligned}
\Gamma^\mu{}_{\gamma\mu}\Gamma^\gamma{}_{tt} &= \Gamma^\mu{}_{t\mu}\Gamma^t{}_{tt} + \Gamma^\mu{}_{r\mu}\Gamma^r{}_{tt} \\
&= (\dot{\phi} + \dot{\lambda} + 2\dot{\chi}/\chi)\dot{\phi} + (\phi' + \lambda' + 2\chi'/\chi)\phi' e^{2\phi-2\lambda} \\
&= \dot{\phi}^2 + \dot{\phi}\dot{\lambda} + 2\dot{\phi}\dot{\chi}/\chi + (\phi'^2 + \phi'\lambda' + 2\phi'\chi'/\chi)e^{2\phi-2\lambda}
\end{aligned}$$

$$\Gamma^\mu{}_{\gamma t}\Gamma^\gamma{}_{t\mu} = \dot{\phi}^2 + \dot{\lambda}^2 + 2\dot{\chi}^2/\chi^2 + 2\phi'^2 e^{2\phi-2\lambda}$$

$$R_{tt} = -\ddot{\lambda} - \dot{\lambda}^2 + \dot{\phi}\dot{\lambda} + 2\dot{\phi}\dot{\chi}/\chi - 2\ddot{\chi}/\chi + (\phi'' + \phi'^2 - \phi'\lambda' + 2\phi'\chi'/\chi)e^{2\phi-2\lambda}$$

(2.2)

$$\begin{aligned}
\partial_\mu \Gamma^\mu{}_{tr} &= \partial_t \Gamma^t{}_{tr} + \partial_r \Gamma^r{}_{tr} \\
&= \partial_t \phi' + \partial_r \dot{\lambda} \\
&= \dot{\phi}' + \dot{\lambda}'
\end{aligned}$$

$$\begin{aligned}
\partial_r \Gamma^\mu{}_{t\mu} &= \partial_r(\dot{\phi} + \dot{\lambda} + 2\dot{\chi}/\chi) \\
&= \dot{\phi}' + \dot{\lambda}' + 2\dot{\chi}'/\chi - 2\dot{\chi}\chi'/\chi^2
\end{aligned}$$

$$\begin{aligned}
\Gamma^\mu{}_{\gamma\mu}\Gamma^\gamma{}_{tr} &= \Gamma^\mu{}_{t\mu}\Gamma^t{}_{tr} + \Gamma^\mu{}_{r\mu}\Gamma^r{}_{tr} \\
&= (\dot{\phi} + \dot{\lambda} + 2\dot{\chi}/\chi)\phi' + (\phi' + \lambda' + 2\chi'/\chi)\dot{\lambda} \\
&= \dot{\phi}\phi' + \phi'\dot{\lambda} + 2\phi'\dot{\chi}/\chi + \phi'\dot{\lambda} + \dot{\lambda}\lambda' + 2\dot{\lambda}\chi'/\chi
\end{aligned}$$

$$\Gamma^\mu{}_{\gamma r}\Gamma^\gamma{}_{t\mu} = \dot{\phi}\phi' + \dot{\lambda}\lambda' + 2\dot{\chi}\chi'/\chi^2 + 2\phi'\dot{\lambda}$$

$$\begin{aligned}
R_{tr} &= 2\phi'\dot{\chi}/\chi + 2\dot{\lambda}\chi'/\chi - 2\dot{\chi}'/\chi \\
&= \frac{2}{\chi}(\phi'\dot{\chi} + \dot{\lambda}\chi' - \dot{\chi}')
\end{aligned}$$

(2.3)

$$\partial_\mu \Gamma^\mu{}_{t\theta} = 0, \quad \partial_\theta \Gamma^\mu{}_{t\mu} = 0, \quad \Gamma^\mu{}_{\gamma\mu}\Gamma^\gamma{}_{t\theta} = \Gamma^\mu{}_{\theta\mu}\Gamma^\theta{}_{t\theta} = (\dot\chi/\chi)\cot\theta, \quad \Gamma^\mu{}_{\gamma\theta}\Gamma^\gamma{}_{t\mu} = (\dot\chi/\chi)\cot\theta$$

$$R_{t\theta} = 0$$

(2.4)

$$\partial_\mu \Gamma^\mu{}_{t\varphi} = 0, \quad \partial_\varphi \Gamma^\mu{}_{t\mu} = 0, \quad \Gamma^\mu{}_{\gamma\mu}\Gamma^\gamma{}_{t\varphi} = 0, \quad \Gamma^\mu{}_{\gamma\varphi}\Gamma^\gamma{}_{t\mu} = 0$$

$$R_{t\varphi} = 0$$

(2.5)

$$\begin{aligned}
\partial_\mu \Gamma^\mu{}_{rr} &= \partial_t \Gamma^t{}_{rr} + \partial_r \Gamma^r{}_{rr} \\
&= \partial_t \dot\lambda e^{2\lambda - 2\phi} + \partial_r \lambda' \\
&= \lambda'' + (\ddot\lambda + 2\dot\lambda^2 - 2\dot\phi\dot\lambda) e^{2\lambda-2\phi}
\end{aligned}$$

$$\begin{aligned}
\partial_r \Gamma^\mu{}_{r\mu} &= \partial_r(\phi' + \lambda' + 2\chi'/\chi) \\
&= \phi'' + \lambda'' + 2\chi''/\chi - 2\chi'^2/\chi^2
\end{aligned}$$

$$\begin{aligned}
\Gamma^\mu{}_{\gamma\mu}\Gamma^\gamma{}_{rr} &= \Gamma^\mu{}_{t\mu}\Gamma^t{}_{rr} + \Gamma^\mu{}_{r\mu}\Gamma^r{}_{rr} \\
&= (\dot\phi + \dot\lambda + 2\dot\chi/\chi)\dot\lambda e^{2\lambda-2\phi} + (\phi' + \lambda' + 2\chi'/\chi)\lambda' \\
&= \phi'\lambda' + \lambda'^2 + 2\lambda'\chi'/\chi + (\dot\lambda^2 + \dot\phi\dot\lambda + 2\dot\lambda\dot\chi/\chi)e^{2\lambda-2\phi}
\end{aligned}$$

$$\Gamma^\mu{}_{\gamma r}\Gamma^\gamma{}_{r\mu} = \phi'^2 + \lambda'^2 + 2\chi'^2/\chi^2 + 2\dot\lambda^2 e^{2\lambda-2\phi}$$

$$R_{rr} = -\phi'' - \phi'^2 + \phi'\lambda' + 2\lambda'\chi'/\chi - 2\chi''/\chi + (\ddot\lambda + \dot\lambda^2 - \dot\phi\dot\lambda + 2\dot\lambda\dot\chi/\chi)e^{2\lambda-2\phi}$$

(2.6)

$$\partial_\mu \Gamma^\mu{}_{r\theta} = 0, \quad \partial_\theta \Gamma^\mu{}_{r\mu} = 0, \quad \Gamma^\mu{}_{\gamma\mu}\Gamma^\gamma{}_{r\theta} = \Gamma^\mu{}_{\theta\mu}\Gamma^\theta{}_{r\theta} = (\chi'/\chi)\cot\theta, \quad \Gamma^\mu{}_{\gamma\theta}\Gamma^\gamma{}_{r\mu} = (\chi'/\chi)\cot\theta$$

$$R_{r\theta} = 0$$

(2.7)

$$\partial_\mu \Gamma^\mu{}_{r\varphi} = 0, \quad \partial_\varphi \Gamma^\mu{}_{r\mu} = 0, \quad \Gamma^\mu{}_{\gamma\mu}\Gamma^\gamma{}_{r\varphi} = 0, \quad \Gamma^\mu{}_{\gamma\varphi}\Gamma^\gamma{}_{r\mu} = 0$$
$$R_{r\varphi} = 0$$

(2.8)

$$\begin{aligned}
\partial_\mu \Gamma^\mu{}_{\theta\theta} &= \partial_t \Gamma^t{}_{\theta\theta} + \partial_r \Gamma^r{}_{\theta\theta} \\
&= \partial_t \dot\chi \chi e^{-2\phi} - \partial_r \chi' \chi e^{-2\lambda} \\
&= (\ddot\chi\chi + \dot\chi^2 - 2\dot\phi\dot\chi\chi)e^{-2\phi} - (\chi''\chi + \chi'^2 - 2\lambda'\chi'\chi)e^{-2\lambda}
\end{aligned}$$

$$\partial_\theta \Gamma^\mu{}_{\theta\mu} = \partial_\theta \cot\theta = -1/\sin^2\theta$$

$$\begin{aligned}
\Gamma^\mu{}_{\gamma\mu}\Gamma^\gamma{}_{\theta\theta} &= \Gamma^\mu{}_{t\mu}\Gamma^t{}_{\theta\theta} + \Gamma^\mu{}_{r\mu}\Gamma^r{}_{\theta\theta} \\
&= (\dot\phi + \dot\lambda + 2\dot\chi/\chi)\dot\chi\chi e^{-2\phi} - (\phi' + \lambda' + 2\chi'/\chi)\chi'\chi e^{-2\lambda} \\
&= (\dot\phi\dot\chi\chi + \dot\lambda\dot\chi\chi + 2\dot\chi^2)e^{-2\phi} - (\phi'\chi'\chi + \lambda'\chi'\chi + 2\chi'^2)e^{-2\lambda}
\end{aligned}$$

$$\Gamma^\mu{}_{\gamma\theta}\Gamma^\gamma{}_{\theta\mu} = 2\dot\chi^2 e^{-2\phi} - 2\chi'^2 e^{-2\lambda} + \cot^2\theta$$

$$\begin{aligned}
R_{\theta\theta} &= (\ddot\chi\chi + \dot\chi^2 - \dot\phi\dot\chi\chi + \dot\lambda\dot\chi\chi)e^{-2\phi} \\
&\quad -(\chi''\chi + \chi'^2 - \lambda'\chi'\chi + \phi'\chi'\chi)e^{-2\lambda} + 1/\sin^2\theta - \cot^2\theta \\
&= (\ddot\chi\chi + \dot\chi^2 - \dot\phi\dot\chi\chi + \dot\lambda\dot\chi\chi)e^{-2\phi} - (\chi''\chi + \chi'^2 - \lambda'\chi'\chi + \phi'\chi'\chi)e^{-2\lambda} + 1
\end{aligned}$$

(2.9)

$$\partial_\mu \Gamma^\mu{}_{\theta\varphi} = 0, \quad \partial_\varphi \Gamma^\mu{}_{\theta\mu} = 0, \quad \Gamma^\mu{}_{\gamma\mu}\Gamma^\gamma{}_{\theta\varphi} = 0, \quad \Gamma^\mu{}_{\gamma\varphi}\Gamma^\gamma{}_{\theta\mu} = 0$$
$$R_{\theta\varphi} = 0$$

(2.10)

$$
\begin{aligned}
\partial_\mu \Gamma^\mu{}_{\varphi\varphi} &= \partial_t \Gamma^t{}_{\varphi\varphi} + \partial_r \Gamma^r{}_{\varphi\varphi} + \partial_\theta \Gamma^\theta{}_{\varphi\varphi} \\
&= \partial_t \dot{\chi}\chi e^{-2\phi} \sin^2\theta - \partial_r \chi'\chi e^{-2\lambda} \sin^2\theta - \partial_\theta \sin\theta\cos\theta \\
&= (\ddot{\chi}\chi + \dot{\chi}^2 - 2\dot{\phi}\dot{\chi}\chi) e^{-2\phi} \sin^2\theta - (\chi''\chi + \chi'^2 - 2\lambda'\chi'\chi) e^{-2\lambda} \sin^2\theta + \sin^2\theta - \cos^2\theta
\end{aligned}
$$

$$\partial_\varphi \Gamma^\mu{}_{\varphi\mu} = 0$$

$$
\begin{aligned}
\Gamma^\mu{}_{\gamma\mu}\Gamma^\gamma{}_{\varphi\varphi} &= \Gamma^\mu{}_{t\mu}\Gamma^t{}_{\varphi\varphi} + \Gamma^\mu{}_{r\mu}\Gamma^r{}_{\varphi\varphi} + \Gamma^\mu{}_{\theta\mu}\Gamma^\theta{}_{\varphi\varphi} \\
&= (\dot{\phi} + \dot{\lambda} + 2\dot{\chi}/\chi)\dot{\chi}\chi e^{-2\phi} \sin^2\theta - (\phi' + \lambda' + 2\chi'/\chi)\chi'\chi e^{-2\lambda} \sin^2\theta - \cot\theta\sin\theta\cos\theta \\
&= (\dot{\phi}\dot{\chi}\chi + \dot{\lambda}\dot{\chi}\chi + 2\dot{\chi}^2) e^{-2\phi} \sin^2\theta - (\phi'\chi'\chi + \lambda'\chi'\chi + 2\chi'^2) e^{-2\lambda} \sin^2\theta - \cos^2\theta
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^\mu{}_{\gamma\varphi}\Gamma^\gamma{}_{\varphi\mu} &= 2\dot{\chi}^2 e^{-2\phi} \sin^2\theta - 2\chi'^2 e^{-2\lambda} \sin^2\theta - 2\cot\theta\sin\theta\cos\theta \\
&= 2\dot{\chi}^2 e^{-2\phi} \sin^2\theta - 2\chi'^2 e^{-2\lambda} \sin^2\theta - 2\cos^2\theta
\end{aligned}
$$

$$
\begin{aligned}
R_{\varphi\varphi} &= (\ddot{\chi}\chi + \dot{\chi}^2 - \dot{\phi}\dot{\chi}\chi + \dot{\lambda}\dot{\chi}\chi) e^{-2\phi} \sin^2\theta - (\chi''\chi + \chi'^2 - \lambda'\chi'\chi + \phi'\chi'\chi) e^{-2\lambda} \sin^2\theta + \sin^2\theta \\
&= R_{\theta\theta} \sin^2\theta
\end{aligned}
$$

(2.11)

$$g_{tt} = -e^{2\phi}, \quad g_{rr} = e^{2\lambda}, \quad g_{\theta\theta} = \chi^2, \quad g_{\varphi\varphi} = \chi^2 \sin^2\theta$$

$$g^{tt} = -e^{-2\phi}, \quad g^{rr} = e^{-2\lambda}, \quad g^{\theta\theta} = 1/\chi^2, \quad g^{\varphi\varphi} = 1/(\chi^2 \sin^2\theta) = g^{\theta\theta}/\sin^2\theta$$

$$
\begin{aligned}
R_{tt} &= -\ddot{\lambda} - \dot{\lambda}^2 + \dot{\phi}\dot{\lambda} + 2\dot{\phi}\dot{\chi}/\chi - 2\ddot{\chi}/\chi + (\phi'' + \phi'^2 - \phi'\lambda' + 2\phi'\chi'/\chi) e^{2\phi - 2\lambda} \\
R_{tr} &= \frac{2}{\chi}(\phi'\dot{\chi} + \dot{\lambda}\chi' - \dot{\chi}') \\
R_{rr} &= -\phi'' - \phi'^2 + \phi'\lambda' + 2\lambda'\chi'/\chi - 2\chi''/\chi + (\ddot{\lambda} + \dot{\lambda}^2 - \dot{\phi}\dot{\lambda} + 2\dot{\lambda}\dot{\chi}/\chi) e^{2\lambda - 2\phi} \\
R_{\theta\theta} &= (\ddot{\chi}\chi + \dot{\chi}^2 - \dot{\phi}\dot{\chi}\chi + \dot{\lambda}\dot{\chi}\chi) e^{-2\phi} - (\chi''\chi + \chi'^2 - \lambda'\chi'\chi + \phi'\chi'\chi) e^{-2\lambda} + 1 \\
R_{\varphi\varphi} &= R_{\theta\theta} \sin^2\theta
\end{aligned}
$$

$$
\begin{aligned}
R &= g^{\alpha\beta} R_{\alpha\beta} \\
&= g^{tt} R_{tt} + g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta} + g^{\varphi\varphi} R_{\varphi\varphi} \\
&= g^{tt} R_{tt} + g^{rr} R_{rr} + 2 g^{\theta\theta} R_{\theta\theta} \\
&= -(-\ddot{\lambda} - \dot{\lambda}^2 + \dot{\phi}\dot{\lambda} + 2\dot{\phi}\dot{\chi}/\chi - 2\ddot{\chi}/\chi) e^{-2\phi} - (\phi'' + \phi'^2 - \phi'\lambda' + 2\phi'\chi'/\chi) e^{-2\lambda} \\
&\quad + (-\phi'' - \phi'^2 + \phi'\lambda' + 2\lambda'\chi'/\chi - 2\chi''/\chi) e^{-2\lambda} + (\ddot{\lambda} + \dot{\lambda}^2 - \dot{\phi}\dot{\lambda} + 2\dot{\lambda}\dot{\chi}/\chi) e^{-2\phi} \\
&\quad + 2(\ddot{\chi}\chi + \dot{\chi}^2 - \dot{\phi}\dot{\chi}\chi + \dot{\lambda}\dot{\chi}\chi) e^{-2\phi}/\chi^2 - 2(\chi''\chi + \chi'^2 - \lambda'\chi'\chi + \phi'\chi'\chi) e^{-2\lambda}/\chi^2 + 2/\chi^2 \\
&= 2[(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\phi}\dot{\lambda} - 2\dot{\phi}\dot{\chi}/\chi + 2\dot{\lambda}\dot{\chi}/\chi + 2\ddot{\chi}/\chi + \dot{\chi}^2/\chi^2) e^{-2\phi} \\
&\quad - (\phi'' + \phi'^2 - \phi'\lambda' + 2\phi'\chi'/\chi - 2\lambda'\chi'/\chi + 2\chi''/\chi + \chi'^2/\chi^2) e^{-2\lambda} + 1/\chi^2]
\end{aligned}
$$

(2.12)

$$G^{\alpha}{}_{\beta} = g^{\alpha\mu}R_{\mu\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}R$$

$$\begin{aligned}
G^{t}{}_{t} &= g^{tt}R_{tt} - (1/2)R \\
&= -(-\ddot{\lambda} - \dot{\lambda}^2 + \dot{\phi}\dot{\lambda} + 2\dot{\phi}\dot{\chi}/\chi - 2\ddot{\chi}/\chi)e^{-2\phi} - (\phi'' + \phi'^2 - \phi'\lambda' + 2\phi'\chi'/\chi)e^{-2\lambda} \\
&\quad -(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\phi}\dot{\lambda} - 2\dot{\phi}\dot{\chi}/\chi + 2\dot{\lambda}\dot{\chi}/\chi + 2\ddot{\chi}/\chi + \dot{\chi}^2/\chi^2)e^{-2\phi} \\
&\quad +(\phi'' + \phi'^2 - \phi'\lambda' + 2\phi'\chi'/\chi - 2\lambda'\chi'/\chi + 2\chi''/\chi + \chi'^2/\chi^2)e^{-2\lambda} - 1/\chi^2 \\
&= -(2\dot{\lambda}\dot{\chi}/\chi + \dot{\chi}^2/\chi^2)e^{-2\phi} + (2\chi''/\chi + \chi'^2/\chi^2 - 2\lambda'\chi'/\chi)e^{-2\lambda} - 1/\chi^2
\end{aligned}$$

$$G^{t}{}_{r} = g^{tt}R_{tr} = -\frac{2e^{-2\phi}}{\chi}(\phi'\dot{\chi} + \dot{\lambda}\chi' - \dot{\chi}')$$

(講義資料は $G^{t}{}_{r}$ の符号が逆)

$$G^{r}{}_{t} = g^{rr}R_{rt} = \frac{2e^{-2\lambda}}{\chi}(\phi'\dot{\chi} + \dot{\lambda}\chi' - \dot{\chi}')$$

$$\begin{aligned}
G^{r}{}_{r} &= g^{rr}R_{rr} - (1/2)R \\
&= (-\phi'' - \phi'^2 + \phi'\lambda' + 2\lambda'\chi'/\chi - 2\chi''/\chi)e^{-2\lambda} + (\ddot{\lambda} + \dot{\lambda}^2 - \dot{\phi}\dot{\lambda} + 2\dot{\lambda}\dot{\chi}/\chi)e^{-2\phi} \\
&\quad -(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\phi}\dot{\lambda} - 2\dot{\phi}\dot{\chi}/\chi + 2\dot{\lambda}\dot{\chi}/\chi + 2\ddot{\chi}/\chi + \dot{\chi}^2/\chi^2)e^{-2\phi} \\
&\quad +(\phi'' + \phi'^2 - \phi'\lambda' + 2\phi'\chi'/\chi - 2\lambda'\chi'/\chi + 2\chi''/\chi + \chi'^2/\chi^2)e^{-2\lambda} - 1/\chi^2 \\
&= (2\phi'\chi'/\chi + \chi'^2/\chi^2)e^{-2\lambda} - (2\ddot{\chi}/\chi + \dot{\chi}^2/\chi^2 - 2\dot{\phi}\dot{\chi}/\chi)e^{-2\phi} - 1/\chi^2
\end{aligned}$$

$$\begin{aligned}
G^{\theta}{}_{\theta} &= g^{\theta\theta}R_{\theta\theta} - (1/2)R \\
&= (\ddot{\chi}/\chi + \dot{\chi}^2/\chi^2 - \dot{\phi}\dot{\chi}/\chi + \dot{\lambda}\dot{\chi}/\chi)e^{-2\phi} - (\chi''/\chi + \chi'^2/\chi^2 - \lambda'\chi'/\chi + \phi'\chi'/\chi)e^{-2\lambda} \\
&\quad +1/\chi^2 - (\ddot{\lambda} + \dot{\lambda}^2 - \dot{\phi}\dot{\lambda} - 2\dot{\phi}\dot{\chi}/\chi + 2\dot{\lambda}\dot{\chi}/\chi + 2\ddot{\chi}/\chi + \dot{\chi}^2/\chi^2)e^{-2\phi} \\
&\quad +(\phi'' + \phi'^2 - \phi'\lambda' + 2\phi'\chi'/\chi - 2\lambda'\chi'/\chi + 2\chi''/\chi + \chi'^2/\chi^2)e^{-2\lambda} - 1/\chi^2 \\
&= -(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\phi}\dot{\lambda} - \dot{\phi}\dot{\chi}/\chi + \ddot{\chi}/\chi + \dot{\lambda}\dot{\chi}/\chi)e^{-2\phi} \\
&\quad +(\phi'' + \phi'^2 - \phi'\lambda' - \lambda'\chi'/\chi + \chi''/\chi + \phi'\chi'/\chi)e^{-2\lambda}
\end{aligned}$$

$$G^{\varphi}{}_{\varphi} = g^{\varphi\varphi}R_{\varphi\varphi} - (1/2)R = g^{\theta\theta}R_{\theta\theta} - (1/2)R = G^{\theta}{}_{\theta}$$